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A Nonlinear Fredholm Integral Equation

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ABSTRACT. This paper contains several results of existence, existence and uniqueness and continuous data dependence of the solution of a nonlinear Fredholm integral equation with modified argument, which appears in the 70′, in some problems from turbo-reactors industry. To obtain these results the Contraction Principle, the Schauder's theorem and the General data dependence theorem are useful.

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1. INTRODUCTION

The integral equations, in general, and those with modified argument, in particular, form an important part of applied mathematics, with links with many theoretical fields, specially with practical fields.

In the study of some problems from turbo-reactors industry, in the 70′, a Fredholm integral equation with modified argument appears, having the following form

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \;\; t \in [a,b], \tag{1′}$$

where $K : [a,b] \times [a,b] \times \mathbb{R}^3 \to \mathbb{R}, \;\; f : [a,b] \to \mathbb{R}$. This integral equation is a mathematical model reference with to the turbo-reactors working.

We study the solution $x^* \in C[a,b]$ of this integral equation and we have obtained the conditions of existence, of existence and uniqueness, of data dependence, of differentiability of the solution with respect to a parameter and of approximation of the solution. These results have been published in the papers [1], [4], [5], [6], [7] and [8].

In the study of the solution of some nonlinear integral equations, in general, and those of type (1), in particular, the papers: V. Berinde [2], D. Guo, V. Lakshmikantham and X. Liu [9], W. Hackbusch [10], V. Mureşan [12], R. Precup [14] and [15], I.A. Rus [16] and [19] and M.A. Şerban [20] have been useful.

In this paper we will extend the results obtained and mentioned above, for the integral equation (1′)

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \;\; t \in [a,b], \tag{1}$$

when $K : [a,b] \times [a,b] \times \mathbb{B}^3 \to \mathbb{B}, \;\; f : [a,b] \to \mathbb{B}$, where $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ is a Banach space.

Our purpose is to study the solution $x^* \in C([a,b], \mathbb{B})$ of this integral equation, in order to establish some results regarding:

- the existence of the solution using the Arzela-Ascoli's theorem and the Schauder's theorem,
- the existence and uniqueness of the solution using the Contraction Principle, and
- the data dependence of the solution using the General data dependence theorem.

2. Notations and preliminaries

Let X be a nonempty set and $A : X \to X$ an operator. In this paper we shall use the following notations:

$P(X) := \{Y \subset X \,/\, Y \neq \varnothing\}$ - the set of all nonempty subsets of X

$A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in N$ - the iterate operators of A

$I(A) := \{Y \in P(X) \,/\, A(Y) \subset Y\}$ - the family of nonempty subsets of X, invariant for A

$F_A := \{x \in X |\ A(x) = x\}$ - the fixed points set of A.

We consider the Banach space $X = C([a,b], \mathbb{B})$ endowed with the Chebyshev norm

$$\|x\|_C := \max_{t\in[a,b]} |x(t)| \ , \quad for\ all \quad x \in C([a,b], \mathbb{B}) \ ,$$

where $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ is a Banach space.

In the section 2, to study the existence and uniqueness of the solution of integral equation (1), we need the following definitions and results (see [3], [11], [13], [17] and [18]).

Let $\{x(t)\}$ be a set of functions $x \in C([a,b], \mathbb{B})$.

Definition 1. *The functions $x(t)$ are called equal bounded functions on the interval $[a,b]$, if there exists $M > 0$ such that*

$$|x(t)| \leq M \quad for\ all\ \ t \in [a,b]\ \ and\ \ x(t) \in \{x(t)\},\ x \in C([a,b], \mathbb{B}).$$

Definition 2. *The functions $x(t)$ are called equal continuous functions on the interval $[a,b]$, if $\forall \varepsilon > 0,\ \ \exists \eta > 0$ such that for each function $x(t) \in \{x(t)\}$, $x \in C([a,b], \mathbb{B})$, we have*

$$|x(t'') - x(t')| \leq \varepsilon \quad for\ all\ \ t'', t' \in [a,b]\ \ and\ \ |t'' - t'| < \eta.$$

Theorem 1. *(Ascoli-Arzela). A subset of the functions from $C([a,b], \mathbb{B})$ is compact if and only if this subset is equal bounded and equal continuous.*

Theorem 2. *(Schauder). Let X be a Banach space and $Y \subset X$ be a nonempty, bounded, convex and closed set. If $A : Y \to Y$ is a completely continuous operator, then A has at least one fixed point.*

Theorem 3. *(Contraction Principle) Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction, ($\alpha < 1$). In these conditions we have:*

(i) $F_A = \{x^*\}$;

(ii) $A^n(x_0) \to x^*$, *as* $n \to \infty$;

(iii) $d(x^*, A^n(x_0)) \le \dfrac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

In order to study the data dependence of the solution of integral equation (1), in the section 3, we need the following result (see [18]).

Theorem 4. *(General data dependence theorem) Let* (X, d) *be a complete metric space and* $A, B : X \to X$ *two operators. We suppose that:*

(i) A *is an* α*-contraction* ($\alpha < 1$) *and* $F_A = \{x_A^*\}$;

(ii) $x_B^* \in F_B$;

(iii) there exists $\eta > 0$ *such that* $d(A(x), B(x)) < \eta$ *for all* $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \le \frac{\eta}{1-\alpha}\,.$$

3. The existence of the solution

In this section we will apply the Contraction Principle and the Schauder's theorem in order to obtain several results of existence and of existence and uniqueness of the solution of integral equation (1). We will study the existence of the solution in the $C([a,b],\mathbb{B})$ space and in the $\overline{B}\,(f;r) \subset C([a,b],\mathbb{B})$ sphere.

A. The existence of the solution in space

We consider the nonlinear Fredholm integral equation with modified argument (1) and assume that the following conditions are satisfied:

(a_1) $K \in C([a,b] \times [a,b] \times \mathbb{B}^3, \mathbb{B})$;

(a_2) $f \in C([a,b], \mathbb{B})$.

In addition, suppose that

(a_3) there exists $M_K > 0$ such that

$$|K(t, s, u_1, u_2, u_3)| \le M_K\,, \quad for\ \ all\ \ t \in [a,b],\ u_1, u_2, u_3 \in \mathbb{B}\,.$$

Theorem 5. *Suppose that the conditions* (a_1)*-*(a_3) *are satisfied. Then the integral equation (1) has at least one solution* $x^* \in C([a,b],\mathbb{B})$.

Proof. We attach to the integral equation (1), the operator $A : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$, defined by

$$A(x)(t) := \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \tag{2}$$

for all $t \in [a,b]$.

Now, using the Chebyshev norm, we obtain

$$\|A(x)\|_C \le \|f\|_C + M_K(b-a)\,, \quad for\ \ all\ \ x \in C([a,b],\mathbb{B})\ .$$

Let $Y \subset C([a,b],\mathbb{B})$ a bounded subset. Then $A(Y)$ is also a bounded subset. From the uniform continuity of K with respect to t, it follows that $A(Y)$ is equal continuous. Therefore, $A(Y)$ is a compact subset.

Let $Y = \overline{conv} A(C([a,b],\mathbb{B}))$.

On the other hand, by the *Ascoli-Arzela's theorem* it results that A is a completely continuous operator.

Since Y is an invariant subset by A, i.e. $Y \in I(A)$, it follows that the conditions of the *Schauder's theorem* are satisfied and the proof is complete. □

Suppose now that the following conditions are satisfied:

(a_4) there exists $L > 0$ such that

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \leq$$
$$\leq L\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$

for all $t, s \in [a,b]$, $u_i, v_i \in \mathbb{B}$, $i = \overline{1,3}$;

(a_5) $3L(b-a) < 1$ and we have the following existence and uniqueness theorem:

Theorem 6. *Suppose that the conditions (a_1)-(a_2) and (a_4)-(a_5) are satisfied. Then the integral equation (1) has a unique solution $x^* \in C([a,b],\mathbb{B})$.*

Proof. We attach to the integral equation (1), the operator $A : C([a,b],\mathbb{B}) \to C([a,b],\mathbb{B})$, defined by the relation (2). The set of the solutions of integral equation (1) coincides with the set of the fixed points of the operator A. By (a_4) and using the Chebyshev norm we have

$$\|A(x_1) - A(x_2)\|_C \leq 3L\,(b-a)\,\|x_1 - x_2\|_C$$

and therefore, by (a_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 3L\,(b-a)$. The proof is complete. □

B. The existence of the solution in sphere

We suppose that

(a_1') $K \in C([a,b] \times [a,b] \times J^3, \mathbb{B})$, $J \subset \mathbb{B}$ is a closed subset,

and also, suppose that the condition (a_2) is satisfied.

In addition, we denote with M_K a positive constant such that, for the restriction $K|_{[a,b]\times[a,b]\times J^3}$, $J \subset \mathbb{B}$ compact, we have

$$|K(t,s,u_1,u_2,u_3)| \leq M_K\,, \quad for\ \ all\ \ t \in [a,b],\ u_1,u_2,u_3 \in J\,. \tag{3}$$

and suppose that the invariability condition of the sphere $\overline{B}(f;r) \subset C([a,b],\mathbb{B})$ is satisfied, i.e.

(b_1) $M_K(b-a) \leq r$.

Theorem 7. *Suppose that the conditions (a_1'), (a_2) and (b_1) are satisfied. Then the integral equation (1) has at least one solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{B})$.*

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;r) \to C([a,b],\mathbb{B})$, defined by the relation (2), where r is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;r)\right] \Longrightarrow [x(t) \in J \subset \mathbb{B}]$$

and suppose that there exists at least one number $r > 0$ with this property.

We establish under what conditions, the sphere $\overline{B}\,(f;r) \subset C([a,b],\mathbb{B})$ is an invariant set for the operator A . We have

$$|A(x)(t) - f(t)| = \left| \int_a^b K(t,s,x(s),x(a),x(b))ds \right| \leq$$

$$\leq \int_a^b |K(t,s,x(s),x(a),x(b))|\, ds$$

and by (3) we have

$$|A(x)(t) - f(t)| \leq M_K(b-a), \quad for \;\; all \;\; t \in [a,b]$$

and then by (b_1) it results that the $\overline{B}(f;r)$ sphere is an invariant set for the operator A, i.e. $\overline{B}(f;r) \in I(A)$. Now, we can consider the operator $A : \overline{B}(f;r) \to \overline{B}(f;r)$, also noted with A, defined by the same relation (2), where $\overline{B}(f;r)$ is a closed subset of the Banach space $C([a,b],\mathbb{B})$.

Next, we assure the conditions of the *Schauder's theorem*.

Since the topology from $\overline{B}(f;r) \subset C([a,b],\mathbb{B})$ is induced of the topology from $C([a,b],\mathbb{B})$, it results that the operator A is continuous.

From $\overline{B}(f;r) \in I(A)$ we have $A(\overline{B}(f;r)) \subset \overline{B}(f;r) \subset C([a,b],\mathbb{B})$ and from $A : \overline{B}(f;r) \to \overline{B}(f;r)$ and using the Chebyshev norm we obtain

$$\|A(x)\|_C \leq \|f\|_C + r\,, \quad for \;\; all \;\; x \in \overline{B}(f;r),$$

that leads to the conclusion that the operator A is equal bounded. From the uniform continuity of the operator A with respect to t it follows that the operator A is equal continuous. By the Ascoli-Arzela's theorem, it results that A is a compact operator. The operator A is continuous and compact and, it follows that A is a completely continuous operator.

The conditions of the *Schauder's theorem* are satisfied and the proof of this theorem is complete. □

Now, suppose that:

(b_2) there exists $L > 0$ such that

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \leq$$
$$\leq L\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right),$$

for all $t,s \in [a,b]$, $u_i, v_i \in J$, $i = \overline{1,3}$,

and also, suppose that the condition (a_5) is satisfied.

We have the following existence and uniqueness theorem:

Theorem 8. *Suppose that the conditions (a_1'), (a_2), (b_1), (b_2) and (a_5) are satisfied. Then the integral equation (1) has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{B})$.*

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;r) \to C([a,b],\mathbb{B})$, defined by the relation (2), where r is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;r)\right] \Longrightarrow [x(t) \in J \subset \mathbb{B}]$$

and suppose that there exists at least one number $r > 0$ with this property.

If we use a reasoning as the one used in the proof of theorem 7, we will obtain that the $\overline{B}(f;r)$ sphere is an invariant set for the operator A, and the invariability condition (b_1), of the $\overline{B}(f;r)$ sphere, is hold.

Now, we can consider the operator $A : \overline{B}(f;r) \to \overline{B}(f;r)$, also noted with A, defined by the same relation, where $\overline{B}(f;r)$ is a closed subset of the Banach space $C([a,b],\mathbb{B})$. The set of the solutions of integral equation (1) coincides with the fixed points set of the operator A.

By a similar reasoning as in the proof of theorem 6 and using the conditions (b_2) and (a_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 3L\,(b-a)$. Therefore, the conditions of the *Contraction Principle* are hold and it results that the operator A has a unique fixed point and consequently, the integral equation (1) has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{B})$. The proof is complete. □

4. THE DATA DEPENDENCE

In what follows our purpose is to establish a result of continuous data dependence of the solution of integral equation (1).

Suppose that the conditions (a_1'), (a_2), (b_1), (b_2) and (a_5) are satisfied. Then by theorem 8 it results that the integral equation (1) has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{B})$.

We attach to the integral equation (1), the operator $A : \overline{B}(f;r) \to C([a,b],\mathbb{B})$, defined by the relation (2). By (b_2) and (a_5) it results that the operator A is an α-contraction with the coefficient $\alpha = 3L(b-a)$ and it follows that the operator A has a unique fixed point, denoted by x_A^*.

The fixed points set of A coincides with the solutions set of integral equation (1). Therefore, the integral equation (1) has the unique solution x_A^*.

Now, we consider the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t)\,, \quad t \in [a,b] \tag{4}$$

and suppose that the following conditions are satisfied:

(d_1) $H \in C([a,b] \times [a,b] \times J^3, \mathbb{B})$, $J \subset \mathbb{B}$ is a closed subset;

(d_2) $h \in C([a,b],\mathbb{B})$;

(d_3) denote with M_H a positive constant such that for the restriction $M_H|_{[a,b]\times[a,b]\times J^3}$, $J \subset \mathbb{B}$ compact, we have:

$$|H(t,s,u_1,u_2,u_3)| \le M_H\,, \quad for\ \ all\ \ t \in [a,b],\ u_1,u_2,u_3 \in J\,;$$

$(d_4) M_H(b-a) \le r$.

Then by the theorem 7 it results that the integral equation (4) has at least one solution in the $\overline{B}(h;r) \subset C([a,b],\mathbb{B})$ sphere.

We attach to the integral equation (4), the operator $B : \overline{B}(h;r) \to \overline{B}(h;r)$, defined by

$$B(y)(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t)\,, \quad t \in [a,b]\,,$$

which has at least one fixed point, denoted by x_B^* . In addition, suppose that

(d_5) there exists $\eta_1 > 0$ such that

$$|K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3)| \leq \eta_1$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3 \in J$;

(d_6) there exists $\eta_2 > 0$ such that

$$|f(t) - h(t)| \leq \eta_2 \quad for \;\; all \quad t, s \in [a,b] \, .$$

By (d_5) and (d_6) we have

$$\|A(x) - B(x)\| \leq \eta_1(b-a) + \eta_2, \quad for \;\; all \;\; x \in \overline{B}(f;r) \subset C([a,b], \mathbb{B}) \, .$$

Therefore, using the General data dependence theorem we obtain the following result:

Theorem 9. *Suppose that the conditions (a_1'), (a_2), (b_1), (b_2) and (a_5) are satisfied and denote with x_A^* the unique solution of the integral equation (1). Moreover, suppose that the conditions (d_1)-(d_5) are satisfied. In these conditions, if x_B^* is a solution of the perturbed integral equation (4), then we have:*

$$\|x_A^* - x_B^*\| \leq \frac{\eta_1(b-a) + \eta_2}{1 - 3L(b-a)} \, .$$

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