TJMM
1 (2009), No. 1-2, 11-23

Distributions from $D'(\mathbb{R}^n)$ Representable only with Respect to the Polar Coordinate $r \in \mathbb{R}$

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ABSTRACT. The notion of distribution from $D'(\mathbb{R}^n)$ representable only with respect to the polar coordinate $r \in \mathbb{R}$ is defined. Some of its properties are proved. It is shown that the distributions $\delta(x),\ \Delta\delta(x) \in D'(\mathbb{R}^n)$ and others are representable with respect to $r \in \mathbb{R}$ and their expressions are given.

2010 *Mathematics Subject Classification.* 46F10, 34A25.

Key words and phrases. *distributions, polar transform*.

1. INTRODUCTION

In the study of some problems from physical-mathematics, sometimes it is useful to change the Cartesian coordinates $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ to the polar coordinates $(r, \theta_1, ..., \theta_{n-1}) \in \mathbb{R}^n$. This necessity leads to the writing of the distributions in the polar coordinates, which be achieved ([1], p.40, [5], p.93) only if the Jacobian of the punctual transformation is different from zero.

Let be $f(x) \in D'(\mathbb{R}^n)$ a distribution with respect to $x \in \mathbb{R}^n$ and $x = h(u),\ h \in C^\infty(\mathbb{R}^n)$ a bijective punctual transformation. The Jacobian of this transformation $J(u) = \dfrac{\partial(x)}{\partial(u)}$ is different from zero. Then, ([1], p.40, [5], p.93) the corresponding distribution $(Tf)(u) \in D'(\mathbb{R}^n)$ with respect to variable $u \in \mathbb{R}$ is given by the formula

$$(f(x),\ \psi(x)) = ((Tf)(u),\ \varphi(u)), \varphi \in D(\mathbb{R}^n)$$

where

$$\psi(u) = \varphi(h(u))\, |J(u)|^{-1}.$$

Consequently, the formula of change of coordinates is inapplicable in the point in which the Jacobian of transformation is zero.

Thus, for the Dirac distribution $\delta(x) \in D'(\mathbb{R}^n)$ concentrated in the origin $(0 \in \mathbb{R}^n)$ point which in polar coordinates is defined by $r = 0$ and $\theta_i \in \mathbb{R},\ i = \overline{1, n-1}$ arbitrary, we can't apply the formula of change of variables because the Jacobian of the transformation is zero. From here it results that the essential polar coordinate in the representation of the distribution $\delta(x) \in D'(\mathbb{R}^n)$ in polar coordinates is $r \in \mathbb{R}$ ([8], p.238).

Thus, the question representation of the distribution $\delta(x) \in D'(\mathbb{R}^n)$ arises, as well as of other distributions from $D'(\mathbb{R}^n)$ with spherical symmetry, with respect to the polar coordinate $r \in \mathbb{R}$.

To solve this problem in [6] we define the functions space $D_p^n(r),\ n \geq 2$ namely

$$D_p^n(r) = \left\{ \psi\ |\psi : \mathbb{R} \to \mathbb{R},\ \ \psi(r) = \int\limits_{S_1(0)} \varphi(rx) dS_1,\ \varphi \in D(\mathbb{R}^n) \right\}$$

where $S_1(0) = \{x,\ \|x\| = 1,\ x \in \mathbb{R}^n\}$, $n \geq 2$ represents the unit sphere centred in the origin, and dS_1 the area element of the sphere.

In connection with the functions space $D_p^n(r)$ the following properties were demonstrated:

1. $D_p^n(r) \subset D(\mathbb{R})$, hence $D_p^n(r)$ is a subspace of the indefinitely derivable functions with compact support $D(\mathbb{R})$;

2. $\psi(-r) = \psi(r),\ r \in \mathbb{R}$;

3. $\psi^{(k)}(0) = 0$ for $k = 1, 3, 5, 7, ...$;

4. $\psi''(0) = \dfrac{S_1}{n} (\Delta\varphi)(0),\ \varphi \in D(\mathbb{R}^n)$, where $|S_1|$ represents the area of the unit sphere from $\mathbb{R}^n$, and Δ the Laplace operator from $\mathbb{R}^n$.

The space $D_p^n(r)$ is named the polar transform of the space $D(\mathbb{R}^n)$, and ψ the polar transform of the function $\varphi \in D(\mathbb{R}^n)$.

We remark that in physics [4], p.45 and in the elasticity theory [9], p.707 the representations of the Dirac distribution with respect to the polar coordinate $r \in \mathbb{R}$ are considered, neither of which being in the framework of the distribution theory. On the other hand, the calculus rules with this representations aren't given.

In this paper the notion of distribution representable only with respect to the polar coordinate $r \in \mathbb{R}$ is given, as well as the polar transformation associated to it. The properties of such distributions are established and it is shown that the distributions $\delta(x),\ \Delta\delta(x) \in D'(\mathbb{R}^n)$ and others are representable with respect to $r \in \mathbb{R}$ and their expressions are given.

2. The test space $D_p^n(r)$, $n \geq 2$ and the polar transformation T_p^n

Let be $f(x) \in D'(\mathbb{R}^n)$ a distribution with respect to the variable $x \in \mathbb{R}^n$ and the punctual transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by the relation $x = h(u)$, where $h \in C^\infty(\mathbb{R}^n)$. Then, (see [1], p.40, [5], p.93), the corresponding distribution with respect to the variable $u \in \mathbb{R}^n$, in the hypothesis that the punctual transformation is bijective, is denoted by $(Tf)(u) \in D'(\mathbb{R}^n)$ and is given by the formula

$$(f(x),\ \psi(x)) = ((Tf)(u),\ \varphi(u)), \varphi \in D(\mathbb{R}^n), \tag{1}$$

where $\psi(x) \in D(\mathbb{R}^n)$ and has the expression

$$\psi(x) = \varphi(u(x))\ \ |J(u)|^{-1}, \tag{2}$$

in which $J(u) = \dfrac{\partial(x)}{\partial(u)}$ represents the Jacobian of the transformation $x = h(u)$.

Thus, the punctual transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, which defines the transformation from the polar coordinates $u = (r, \theta_1, \theta_2, ..., \theta_{n-1}) \in \mathbb{R}^n$ to the Cartesian coordinates $x = (x_1, ..., x_n) \in \mathbb{R}^n$, is defined by the relations

$$\begin{array}{l} x_1 = r\sin\theta_1 \sin\theta_2 ... \sin\theta_{n-2}\sin\theta_{n-1}, \\ x_2 = r\sin\theta_1 \sin\theta_2 ... \sin\theta_{n-2}\cos\theta_{n-1}, \\ x_3 = r\sin\theta_1 \sin\theta_2 ... \sin\theta_{n-3}\cos\theta_{n-2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-2} = r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_{n-1} = r\sin\theta_1\cos\theta_2, \\ x_n = r\cos\theta_1. \end{array} \tag{3}$$

The Jacobian of the transformation being

$$J(u) = J(r,\theta_1,...,\theta_{n-1}) = \frac{\partial(x)}{\partial(u)} = r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 ... \sin\theta_{n-2}. \tag{4}$$

To the transformation $u \to x$, defined by (3), we impose generally the restrictions

$$r \geq 0, \theta_i \in [0,\pi], i = \overline{1, n-2}, \theta_{n-1} \in [0, 2\pi), \tag{5}$$

which do not ensure the bijection of the transformation everywhere.

For example, the point $0 \in \mathbb{R}^n$ which represents the origin of the Cartesian reference system is a singular point of the punctual transformation (3), which in polar coordinates is defined by $r = 0$ and $\theta_i \in \mathbb{R}$, $i = \overline{1, n-1}$, arbitrary. The Jacobian of the transformation is null in this point.

Consequently, the formulas (1) and (2) become inapplicable in the point in which the Jacobian of the punctual transformation (3) is zero.

Let be, for example, the Dirac distribution $\delta_{x_0} = \delta(x - x_0) \in D'(\mathbb{R}^n)$ concentrated in the point $x_0 \in \mathbb{R}^n$. Let be $u_0 = (r_0, \theta_1^0, ..., \theta_{n-1}^0) \in \mathbb{R}^n$ the corresponding point of $x_0 \in \mathbb{R}^n$ to the transformation (3) with the restrictions (5) and isn't be singular point of this transformation, hence $J(u_0) \neq 0$.

Although the punctual transformation (3) is not bijective everywhere, the formulas (1) and (2) can be applied in the case of the Dirac distribution and we have

$$(\delta(x - x_0),\ \psi(x)) = ((T\delta_{x_0})(u),\ \varphi(u)) = \psi(x_0), \tag{6}$$

$$\psi(x_0) = \varphi(u_0)\ |J(u_0)|^{-1}. \tag{7}$$

From these relations we obtain

$$((T\delta_{x_0})(u),\ \varphi(u)) = \frac{\varphi(u_0)}{|J(u_0)|} = \left(\frac{\delta(u-u_0)}{|J(u_0)|},\ \varphi(u)\right), \varphi \in D(\mathbb{R}^n),$$

wherefrom

$$(T\delta_{x_0})(u) = \frac{\delta_{u_0}}{|J(u_0)|} = \frac{\delta(r - r_0, \theta_1 - \theta_1^0, ..., \theta_{n-1} - \theta_{n-1}^0)}{r_0^{n-1}\sin^{n-2}\theta_1^0 \sin^{n-3}\theta_2^0 ... \sin\theta_{n-2}^0}. \tag{8}$$

The expression (8) represents the Dirac distribution in polar coordinates and this take places in the points in which the Jacobian of the transformation isn't zero.

Thus, in $\mathbb{R}^2$ the change from the polar coordinates (r, θ) to the Cartesian coordinates is defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $(r, \theta) \in \mathbb{R}^2$, the Jacobian of the transformation being $J(r, \theta) = r$.

Hence, the only singular point of this transformation is $x = 0$, $y = 0$, which in polar coordinates is defined by $r = 0$ and $\theta \in \mathbb{R}$, arbitrary.

In order that the transformation should be bijective everywhere with the exception of the origin of the Cartesian reference system we impose the restrictions $r > 0$, $\theta \in [0, 2\pi)$.

Applying the formula (8) for distribution $\delta(x - x_0,\ y - y_0) \in\ D'(\mathbb{R}^2$, where $(x_0, y_0) \neq (0, 0)$ we obtain

$$T\left(\delta(x - x_0, y - y_0)\right)(r, \theta) = \frac{\delta(r - r_0, \theta - \theta_0)}{r_0} = \\ = \frac{1}{r_0}\delta(r - r_0) \times \delta(\theta - \theta_0), \tag{9}$$

where $x_0 = r_0 \cos\theta_0$, $y_0 = r_0 \sin\theta_0$, $r_0 > 0$, $\theta_0 \in [0, 2\pi)$.

Hence, the Dirac distribution $\delta(x - x_0,\ y - y_0)$ in polar coordinates has the expression (9).

Analogously, in $\mathbb{R}^3$ the application $(r, \theta_1, \theta_2) \to (x, y, z)$ defined by

$$x = r\sin\theta_1\cos\theta_2, y = r\sin\theta_1\sin\theta_2, z = r\cos\theta_1, \tag{10}$$

where $r \geq 0$, $\theta_1 \in [0, \pi]$, $\theta_2 \in [0, 2\pi)$, represents the formulas of changing from the Cartesian coordinates to polar coordinates, having the Jacobian $J(r, \theta_1, \theta_2) = r^2 \sin\theta_1$.

This transformation is bijective everywhere with the exception of the point from the Oz-axis, for which the Jacobian is zero.

Using the formula (8) we obtain

$$\left(T\delta(x - x_0, y - y_0, z - z_0)\right)(r, \theta_1, \theta_2) = \frac{\delta(r - r_0, \theta_1 - \theta_1^0, \theta_2 - \theta_2^0)}{r_0^2 \sin\theta_1^0}, \tag{11}$$

where $x_0 = r_0 \sin\theta_1^0 \cos\theta_2^0$, $y_0 = r_0 \sin\theta_1^0 \sin\theta_2^0$, $z_0 = r_0 \sin\theta_1^0$ and obviously $r_0^2 \sin\theta_1^0 \neq 0$.

According to the above mentioned, the formulas (8), (9) and (11) are inapplicable in the points in which the Jacobian of the transformation is null.

The problem arises if, for the Dirac distribution concentrated in a singular point of the transformation, we can define a representation in polar coordinates and what would be the natural introduction of this representation.

We remark that in physics ([5], p.45) in the solving of some problems from electrodynamics,or in quantum mechanics, for the Dirac distribution $\delta(x, y) \in D'(\mathbb{R}^2)$ in polar coordinates the representation $\dfrac{\delta(r)}{2\pi r}$ is used.

Also, for $\delta(x, y, z) \in D'(\mathbb{R}^3)$ in polar coordinates, the representation $\dfrac{\delta(r)}{4\pi r^2}$ is used.

We notice that these representations in polar coordinates given to the Dirac distribution have no meaning from the point of view of the distribution theory, because doesn't exists the products between the Dirac distribution $\delta(r)$ and the functions with singularity in origin $\dfrac{1}{r}$ and $\dfrac{1}{r^2}$.

In [2], p.290, is approached the problem of the representation of the Dirac distribution in polar coordinates, but not in the distributions meaning, and utilization rules are not given (doesn't exists).

To justify the notions what follows to be introduced we shall analyse the representation in polar coordinates of the Dirac distributions $\delta(x, y) \in D'(\mathbb{R}^2)$.

This distribution has as support the origin of the Euclidean space $\mathbb{R}^2$ which represents a singular point of the transformation: $x = r\cos\theta$, $y = r\sin\theta$, $r \geq 0$, $\theta \in [0, 2\pi)$.

The singular point $(0,0) \in \mathbb{R}^2$ in polar coordinates is defined by $r = 0$ and $\theta \in \mathbb{R}$ arbitrary.

Hence, the essential polar coordinate in the representation of the distribution $\delta(x,y)$ in polar coordinates is $r \in \mathbb{R}$ ([8], p.238). From this reason it is natural to admit that the representation in polar coordinates of the distribution $\delta(x,y)$ will depend only by the variable $r \in \mathbb{R}$.

Using the formalism of changing of variables to the integrals and using the polar coordinates we have

$$\begin{gathered}(\delta(x,y),\ \varphi(x,y)) = \int\limits_{\mathbb{R}^2} \delta(x,y)\varphi(x,y)\mathrm{d}x\mathrm{d}y = \\ = \int\limits_0^\infty \int\limits_0^{2\pi} \tilde{\delta}(r)\tilde{\varphi}(r,\theta) r \mathrm{d}r\mathrm{d}\theta = \int\limits_0^\infty \left(r\tilde{\delta}(r) \int\limits_0^{2\pi} \tilde{\varphi}(r,\theta)\mathrm{d}\theta \right) \mathrm{d}r,\end{gathered} \tag{12}$$

where $\varphi \in D(\mathbb{R}^2)$, and $\tilde{\delta}(r)$ represents the distribution $\delta(x,y)$ in polar coordinates, $\tilde{\varphi}(r,\theta) = \varphi(r\cos\theta, r\sin\theta)$.

In the distribution notation the relation (12) can be written under the form

$$(\delta(x,y),\ \varphi(x,y)) = \left(rH(r)\tilde{\delta}(r),\ \psi(r) \right), \varphi \in D(\mathbb{R}^2), \tag{13}$$

where

$$\psi(r) = \int\limits_0^{2\pi} \tilde{\varphi}(r,\theta)\mathrm{d}\theta. \tag{14}$$

Taking into account the space $D_p^n(r)$ introduce in [6], we observe that ψ given by (14) is an element of this space for $n = 2$, and $rH(r)\tilde{\delta}(r) = F(r)$ is a functional defined on the space $D_p^2(r)$, where $H(r),\ r \in \mathbb{R}$ represents the Heaviside functions.

This space is named the polar transform of the space $D(\mathbb{R}^2)$ and will be note $T_p^2\left(D(\mathbb{R}^2)\right)$.

Next we introduce:

Definition 1. *The distribution $f \in D'(\mathbb{R}^n)$ is representable in the polar coordinate $r \in \mathbb{R}$, if exists the functional $F : D_p^n(r) \to \mathbb{C}$ with the property*

$$(f,\ \varphi) = (F,\ \psi)\,, (\forall)\,\varphi \in D(\mathbb{R}^n), \tag{15}$$

where

$$\psi(r) \in D_p^n(r), \psi(r) = \int_{S_1(0)} \varphi(rx)\mathrm{d}S_1, \tag{16}$$

and $S_1(0)$ represents the unit sphere centred in the origin, and dS_1 the area element of the sphere.

Taking into account [6], the function $\psi \in D_p^n(r)$ represents the polar transform with respect to $r \in \mathbb{R}$ of the function $\varphi \in D(\mathbb{R}^n)$ and will be writing $\psi = T_p^n(\varphi)$.

Proposition 1. *Let be $f \in D'(\mathbb{R}^n)$ a distribution representable with respect to the polar coordinate $r \in \mathbb{R}$. Then the functional $F : D_p^n(r) \to C$ defined by (15) is unique, linear and continuous, hence $F \in D_p^{n\prime}(r) \subset D'(R)$.*

Proof. Let be $\psi_1,\ \psi_2 \in D_p^n(r)$ and $\psi = \alpha\psi_1 + \beta\psi_2$, $\alpha, \beta \in \mathbb{C}$, where $\psi_1 = \int\limits_{S_1(0)} \varphi_1(rx)\mathrm{d}S_1$, $\psi_2 = \int\limits_{S_1(0)} \varphi_2(rx)\mathrm{d}S_1$, $\varphi_1,\ \varphi_2 \in D(\mathbb{R}^n)$. We denote $\varphi = \alpha\varphi_1 + \beta\varphi_2 \in D(\mathbb{R}^n)$. Then, according to the definition, we can write $(f,\ \varphi_1) = (F,\ \psi_1)$, $(f,\ \varphi_2) = (F,\ \psi_2)$. Thus, we have

$$(F,\ \psi) = (F,\ \alpha\psi_1 + \beta\psi_2) = (f,\ \alpha\varphi_1 + \beta\varphi_2)\,. \tag{17}$$

Due to the linearity of the functional $f \in D'(\mathbb{R}^n)$ the above relation becomes

$$(F,\ \psi) = \alpha\,(f,\ \varphi_1) + \beta\,(f,\ \varphi_2) = \alpha\,(F,\ \psi_1) + \beta\,(F,\ \psi_2)\,,$$

which show the linearity of the functional F.

Concerning the continuity of the functional F we consider the sequence $(\varphi_k(x)) \to 0$ in $D(\mathbb{R}^n)$ whom in $D_p^n(r)$ it corresponds the sequence

$$(\psi_k(r))\,, \psi_k(r) = \int_{S_1(0)} \varphi_k(rx)\mathrm{d}S_1, \varphi_k \in D(\mathbb{R}^n). \tag{18}$$

Since we have $(f,\ \varphi_k) = (F,\ \psi_k)$, and $\lim\limits_k (f,\ \varphi_k) = 0$ because $(\varphi_k) \to 0$ in $D(\mathbb{R}^n)$ from (18) we obtain

$$\lim_k\,(F,\ \psi_k) = 0. \tag{19}$$

Taking into account [6], the polar transform T_p^n is a continuous operator from $D(\mathbb{R}^n)$ in $D_p^n(r) = T_p(D(\mathbb{R}^n))$, hence if $(\varphi_k) \overset{D(\mathbb{R}^n)}{\longrightarrow} 0$, then this implies $(\psi_k) = \left(T_p^n(\varphi_k)\right) \overset{D_p^n(r)}{\longrightarrow} 0 = T_p^n(0)$.

Consequently, because $(\psi_k) \to 0$ in $D_p^n(r) \subset D(\mathbb{R})$ and according to (19) $\lim\limits_k (F,\ \psi_k) = 0$ which it mean that the functional F is continuous on $D_p^n(r)$, so F will be named distribution from $D_p^{n\prime}(r) \subset D'(\mathbb{R})$.

The space of the distributions $D_p^{n\prime}(r) \subset D'(\mathbb{R})$ represents the set of the linear and continuous functionals on the space $D_p^n(r) \subset D(\mathbb{R})$.

Concerning the unicity of the distribution $F \in {D'_p}^n(r)$ this results from the equality $(f,\ \varphi) = (F,\ \psi) = (F_1,\ \psi)$, $F,\ F_1 \in {D'_p}^n(r)$, $\psi \in D_p^n(r)$, wherefrom $(F - F_1,\ \psi) = 0$, $(\forall)\,\psi \in D_p^n(r)$ which implies $F = F$. With this the proposition is proved. □

Definition 2. *Let $f \in D'(\mathbb{R}^n)$ be a distribution representable with respect to the polar coordinate $r \in \mathbb{R}$, then the distribution $F \in D_p^{n\prime}(r)$ defined by the relation (15) is named the polar transform of the distribution $f \in D'(\mathbb{R}^n)$ and will be note $F(r) = T_p^n f$.*

With this the equality (15) becomes

$$(f,\ \varphi) = \left(T_p^n f,\ T_p^n \varphi\right), T_p^n \varphi = \psi \in D_p^n(r), \tag{20}$$

and it represents the definition relation for the distributions from $D'(\mathbb{R}^n)$ which admit polar transformation, hence its are representable with respect to the polar coordinate $r \in \mathbb{R}$.

If we denote by $D'_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ the subset of the distributions from $D'(\mathbb{R}^n)$ which admit polar transformation, then we can say that the polar transform T_p^n is an operator which applies the subspace $D'_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ in the space $D_p^{n\prime}(r) \subset D'(\mathbb{R})$ according to the formula (20).

Proposition 2. *The polar transformation $T_p^n : D'_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n) \to D_p^{n\prime}(r) \subset D'(\mathbb{R})$, defined by (20) represent a linear and continuous operator from $D'_p(\mathbb{R}^n)$ to ${D'_p}^n(r)$.*

Proof. Let be $f, g \in D'_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ and $T_p^n f, T_p^n g \in D_p^{n\prime}(r) \subset D'(\mathbb{R})$. Then $\forall \alpha, \beta \in \mathbb{C}$, $\forall \varphi \in D(\mathbb{R}^n)$ we have

$$(\alpha f + \beta g,\ \varphi) = \alpha\,(f,\ \varphi) + \beta\,(g,\ \varphi) = \alpha\left(T_p^n f,\ T_p^n \varphi\right) + \beta\left(T_p^n g,\ T_p^n \varphi\right), \qquad (21)$$

because $(f,\ \varphi) = \left(T_p^n f,\ T_p^n \varphi\right)$, $(g,\ \varphi) = \left(T_p^n g,\ T_p^n \varphi\right)$.

Taking into account $T_p^n f$, $T_p^n g$ are distributions from ${D'_p}^n(r) \subset D'(\mathbb{R})$ and $T_p^n \varphi \in D_p^n(r) \subset D(\mathbb{R})$, the relation relaţia (21) becomes

$$(\alpha f + \beta g,\ \varphi) = \left(\alpha T_p^n f + \beta T_p^n g,\ T_p^n \varphi\right),$$

and according to the definition 1 this mean that exists $T_p^n(\alpha f + \beta g)$ and we have $T_p^n(\alpha f + \beta g) = \alpha T_p^n f + \beta T_p^n g$.

We shall show the continuity of the operator T_p^n. Let be $f_k \in D'_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ and $(f_k) \to 0$, $k \to \infty$ in $D'_p(\mathbb{R}^n)$, hence in $D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$. Using (20) we have $\left(T_p^n f_k,\ T_p^n \varphi\right) = (f_k,\ \varphi) \to 0$, $k \to \infty$, wherefrom it results $T_p^n f_k \to 0$, $k \to \infty$ in ${D'_p}^n(r)$. The proposition is thus proved. □

We introduce the functional $\delta : D_p^n(r) \to \mathbb{R}$ by the formula

$$(\delta(r),\ \psi(r)) = \psi(0), \psi \in D_p^n(r). \qquad (22)$$

Obviously, the functional $\delta(r)$ is linear and continuous on $D_p^n(r) \subset D(\mathbb{R})$, hence is a distribution from ${D'_p}^n(r)$.

On the base of the definition of the distribution $\delta(r) \in {D'_p}^n(r)$ we have

$$\delta^{(2k-1)}(r) = 0, k = 1, 2, 3, ... \qquad (23)$$

Indeed, taking into account [6] we have $\psi^{(k)}(0) = 0$, $k = 1, 3, 5, ...$, hence $\psi^{(2k-1)}(0) = 0$, $k = 1, 2, 3, ...$.

Consequently, it yields

$$\left(\delta^{(2k-1)}(r),\ \psi(r)\right) = (-1)^{2k-1}\left(\delta(r),\ \psi^{(2k-1)}(r)\right) = -\psi^{(2k-1)}(0) = 0,$$

hence the relation (23).

Proposition 3. *The Dirac distribution $\delta(x) \in D'(\mathbb{R}^n)$ admits polar transformation and its expression is*

$$T_p^n \delta(x) = \frac{\delta(r)}{|S_1|} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \delta(r), n \geq 2, \tag{24}$$

where $|S_1| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ *represents the area of the unit sphere from* $\mathbb{R}^n$.

Proof. For any $\varphi \in D(\mathbb{R}^n)$ we have

$$(\delta(x),\ \varphi(x)) = \varphi(0). \tag{25}$$

On the other hand, from (16) we obtain

$$\psi(0) = \int\limits_{S_1(0)} \varphi(0) \mathrm{d}S_1 = \varphi(0) |S_1|, \tag{26}$$

hence $\varphi(0) = \frac{\psi(0)}{|S_1|}$.

Taking into account (22), the relation (25) becomes

$$(\delta(x),\ \varphi(x)) = \frac{\psi(0)}{|S_1|} = \left(\frac{\delta(r)}{|S_1|},\ \psi(r) \right) = \left(T_p^n \delta(x),\ T_p^n \varphi(x) \right),$$

wherefrom on the basis of the formulae (20) we obtain the relation (24). □

Particularly, for $n = 2$ and $n = 3$ we obtain the polar transformations

$$T_p^2 \delta(x, y) = \frac{\delta(r)}{2\pi}, |S_1| = 2\pi,$$
$$T_p^3 \delta(x, y, z) = \frac{\delta(r)}{4\pi}, |S_1| = 4\pi.$$

We define now the functional $\frac{\delta(r)}{r} : D_p^n(r) \to \mathbb{R}$ by the formula

$$\left(\frac{\delta(r)}{r},\ \psi(r) \right) = \mathrm{VP} \int\limits_R \frac{\psi(r) - \psi(0)}{r} \mathrm{d}r. \tag{27}$$

We shall show that the functional $\frac{\delta(r)}{r}$ represents a distribution of first order from ${D'_p}^n(r)$.

Obviously, the functional $\frac{\delta(r)}{r}$ is linear, and its continuity results from the inequality

$$\left| \left(\frac{\delta(r)}{r},\ \psi(r) \right) \right| \leq \mathrm{VP} \int\limits_{-a}^{a} \left| \frac{\psi(r) - \psi(0)}{r} \right| \mathrm{d}r \leq 2a \sup_{r \in [-a,a]} |\psi'(r)|,$$

where $\operatorname{supp} \psi \subset [-a, a]$.

Proposition 4. *The functional* $\frac{\delta(r)}{r^2} : D_p^n(r) \to \mathbb{R}$ *defined by the formulae*

$$\left(\frac{\delta(r)}{r^2},\ \psi(r) \right) = \mathrm{VP} \int\limits_R \frac{\psi(r) - \psi(0)}{r^2} \mathrm{d}r, \psi \in D_p^n(r), \tag{28}$$

represents a distribution of second order from ${D'_p}^n(r)$.

Proof. Because the linearity of the functional is obviously, we shall show only its continuity. Taking into account $\psi(r) - \psi(0) = r\psi'(0) + +\frac{r^2}{2}\psi''(\xi)$, $\xi \in (0, r)$ or $\xi \in (r, 0)$ as well as of $\psi'(0) = 0$ we obtain

$$\left|\left(\frac{\delta(r)}{r^2},\ \psi(r)\right)\right| = \frac{1}{2}\left|\int\limits_{-a}^{a} \psi''(\xi)\mathrm{d}r\right| \leq a \sup_{r\in[-a,a]} |\psi''(r)|\,, \tag{29}$$

where $\operatorname{supp}\psi \subset [-a, a]$, $a > 0$.

The relation (29) proves the continuity of the functional $\frac{\delta(r)}{r^2}$ as well as the fact that its a distribution of second order from ${D'_p}^n(r)$. □

For $n = 2$, using the formula (27), we obtain the distribution

$$F(r) = \frac{\delta(r)}{2\pi r} \in {D'_p}^2(r), \tag{30}$$

which in the elasticity theory [9], p.707, and in quantic mechanics [4], p.45, it is considered the corresponding of the Dirac distribution $\delta(x, y) \in D'(\mathbb{R}^2)$ in polar coordinates.

Analogously, for $n = 3$, the distribution

$$G(r) = \frac{\delta(r)}{4\pi\ r^2} \in {D'_p}^3(r), \tag{31}$$

obtained with the help of the formula (28), is considered the corresponding of the Dirac distribution $\delta(x, y, z) \in D'(\mathbb{R}^3)$ by passing to spherical coordinates.

We remark that the use of these distributions in mechanics and physics in general do not observe the rules of theory of distributions. Therefore, formulas (27), (28) justify their operation in terms of the theory of distributions.

Proposition 5. *The distribution $\Delta\delta(x) \in D'(\mathbb{R}^n)$,where Δ represents the Laplace operator, admits polar transformation and we have*

$$T_p^n\left(\Delta\delta(x)\right) = \frac{n}{|S_1|}\delta''(r), \quad |S_1| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, n \geq 2. \tag{32}$$

Proof. For any $\varphi \in D(\mathbb{R}^n)$ we have

$$(\Delta\delta(x),\ \varphi(x)) = (\delta(x),\ \Delta\varphi(x)) = (\Delta\varphi)\,(0).$$

Taking into account the relation $\psi''(0) = \frac{|S_1|}{n}(\Delta\varphi)(0)$ obtained in [6] we have

$$(\Delta\delta(x),\ \varphi(x)) = (\Delta\varphi)\,(0) = \frac{n}{|S_1|}\psi''(0) = \left(\frac{n}{|S_1|}\delta''(r),\ \psi(r)\right),$$

wherefrom it results the formula (32). □

Particularly, for $n = 2$ and $n = 3$ from (32) we obtain

$$\begin{array}{c} T_p^2\left(\Delta\delta(x,y)\right) = \frac{1}{\pi}\delta''(r), \quad |S_1| = 2\pi, \\ T_p^3\left(\Delta\delta(x,y,z)\right) = \frac{3}{4\pi}\delta''(r), \quad |S_1| = 4\pi. \end{array} \tag{33}$$

Proposition 6. *Let be $f \in D'(\mathbb{R}^n)$ and $f_\varepsilon \in \mathrm{L}_{loc}(\mathbb{R}^n)$, $\varepsilon > 0$ a family of locally integrable functions such that $\lim\limits_{\varepsilon \to +0} f_\varepsilon(x) = f(x)$ in $D'(\mathbb{R}^n)$. If f_ε is a function with spherical symmetry, hence $f_\varepsilon(x) = f_\varepsilon^*(r) = f_\varepsilon^*(\|x\|)$, then the distribution $f \in D'(R^n)$ admits the polar transformation $T^n f \in {D'_p}^n(r)$ and we have*

$$T_p^n f(x) = \lim_{\varepsilon \to +0} r^{n-1} f_\varepsilon^*(r) H(r) = F(r), \tag{34}$$

where $H(r)$ represents the Heaviside function.

Proof. We have

$$\lim_{\varepsilon \to +0} (f_\varepsilon(x),\ \varphi(x)) = (f(x),\ \varphi(x)), \forall \varphi \in D(\mathbb{R}^n). \tag{35}$$

Passing to the polar coordinates, the relation (35) becomes

$$(f,\ \varphi) =$$
$$\lim_{\varepsilon \to +0} \int_0^\infty \int_0^\pi ... \int_0^\pi \int_0^{2\pi} f_\varepsilon^*(r) \varphi^*(r, \theta_1, \theta_2, \ldots, \theta_{n-1}) \left| J(r, \theta_1, \ldots, \theta_{n-1}) \right| \mathrm{d}r \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{n-1}$$

Taking into account (16) we obtain

$$(f,\ \varphi) = \lim_{\varepsilon \to +0} \int_0^\infty r^{n-1} f_\varepsilon^*(r) \left\{ \int_{S_1(0)} \varphi(x) \mathrm{d}S_1 \right\} \mathrm{d}r =$$
$$= \lim_{\varepsilon \to +0} \int_{-\infty}^\infty F_\varepsilon(r) H(r) \psi(r) \mathrm{d}r = \lim_{\varepsilon \to +0} (F_\varepsilon(r) H(r),\ \psi(r)),$$

where $F_\varepsilon(r) = r^{n-1} f_\varepsilon^*(r)$.

For here it results that the distribution $F(r) \in {D'_p}^n(r)$ exists such that

$$(f,\ \varphi) = \lim_{\varepsilon \to +0} (F_\varepsilon(r) H(r),\ \psi(r)) = (F(r),\ \psi(r)),$$

as well as it certifies that the distribution $f \in D'(\mathbb{R}^n)$ has the polar transformation $T_p^n f(x) = F(r)$, and its expression is given by (34). The proposition is thus proved. □

We remark that result gives us a sufficient condition so that the distribution $f \in D'(\mathbb{R}^n)$ to admit polar transformation.

Particularly, if $f(x) = f^*(r) \in \mathrm{L}_{loc}(\mathbb{R}^n)$, and $f(x) = f^*(r) = f_\varepsilon^*(r)$ then the function type distribution $f(x)$ admits polar transformation whose expression is

$$T_p^n f(x) = r^{n-1} f^*(r) H(r) = F(r),\ r \in \mathbb{R}. \tag{36}$$

Example 1. *Let be the functions family $f_\varepsilon(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$ Dirac representable, $\lim\limits_{\varepsilon \to +0} f_\varepsilon(x) = \delta(x)$ ([5], p.164),*

$$f_\varepsilon(x) = \frac{n}{|S_1|} \frac{\varepsilon^2}{\left(\varepsilon^2 + \|x\|^2\right)^{\frac{n+2}{2}}},\ n \geq 2.$$

We observe that the conditions from the proposition 6 are fulfill, because $f_\varepsilon \in \mathrm{L}_{loc}(\mathbb{R}^n)$, *and* $f_\varepsilon(x) = f_\varepsilon^*(r) = \frac{n}{|S_1|} \frac{\varepsilon^2}{(\varepsilon^2+r^2)^{\frac{n+2}{2}}}$.

Consequently, the polar transformation $T_p^n\delta(x) \in D_p'^{\,n}(r) \subset D'(\mathbb{R})$ *exists and we have*

$$T_p^n\delta(x) = \lim_{\varepsilon\to+0} r^{n-1}H(r)\frac{n}{|S_1|}\,\frac{\varepsilon^2}{(\varepsilon^2+r^2)^{\frac{n+2}{2}}}.$$

Taking into account (24) we obtain

$$\lim_{\varepsilon\to+0} r^{n-1}H(r)\,\frac{n\varepsilon^2}{(\varepsilon^2+r^2)^{\frac{n+2}{2}}} = \delta(r), r \in \mathbb{R}. \tag{37}$$

Particularly, for $n = 2$ *we have*

$$\lim_{\varepsilon\to+0} rH(r)\,\frac{2\varepsilon^2}{(\varepsilon^2+r^2)^2} = \delta(r), r \in \mathbb{R}. \tag{38}$$

Example 2. *Let be the function* $f(x) = \frac{1}{(\sqrt{\pi})^n}\mathrm{e}^{-\|x\|^2}$, $x \in \mathbb{R}^n$. *Because* $f \in \mathrm{L}_{loc}(\mathbb{R}^n)$ *and* $f(x) = f^*(r) = \frac{1}{(\sqrt{\pi})^n}\mathrm{e}^{-r^2}$, $r \in \mathbb{R}$, *then, according to the formula (36),* $T_p^n f(x)$ *exists and we have*

$$T_p^n f(x) = r^{n-1}f^*(r)H(r) = r^{n-1}\frac{\mathrm{e}^{-r^2}H(r)}{(\sqrt{\pi})^n},\ r \in \mathbb{R},\ n \geq 2. \tag{39}$$

Taking into account ([5], p.166) we have

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\frac{1}{(\sqrt{\pi})^n}\mathrm{e}^{-\|x\|^2/\varepsilon^2} \overset{D'(\mathbb{R}^n)}{\longrightarrow} \delta(x),\ x \in \mathbb{R}^n,\ \varepsilon > 0.$$

According to the proposition 6 and the formula (24) we obtain

$$T_p^n\delta(x) = \lim_{\varepsilon\to+0} r^{n-1}\frac{1}{\varepsilon^n}\frac{1}{(\sqrt{\pi})^n}\mathrm{e}^{-r^2/\varepsilon^2}H(r) = \frac{\delta(r)}{|S_1|},$$

wherefrom it results

$$\lim_{\varepsilon\to+0} |S_1|\, r^{n-1}\frac{1}{\varepsilon^n}\frac{1}{(\sqrt{\pi})^n}\mathrm{e}^{-r^2/\varepsilon^2}H(r) = \delta(r), r \in \mathbb{R}, n \geq 2. \tag{40}$$

Proposition 7. *Let be the distributions* $F(r) \in D_p'^{\,n}(r)$, $F_\varepsilon(r) \in \mathrm{L}_{loc}(\mathbb{R})$, $\varepsilon > 0$ *and* $F_\varepsilon(r) \underset{\varepsilon\to0}{\overset{D_p'^{\,n}}{\longrightarrow}} F(r)$. *If the function* $\mu(r) \in \mathrm{C}^\infty(\mathbb{R})$ *is a multiplier of the space* $D_p^n(r)$, *then we have*

$$\mu(r)F_\varepsilon(r) \underset{\varepsilon\to0}{\overset{D_p'^{\,n}}{\longrightarrow}} \mu(r)F(r). \tag{41}$$

Proof. Because $\mu(r)$ is a multiplier of the space $D_p^n(r)$ we have

$$(F_\varepsilon(r),\ \mu(r)\psi(r)) = (\mu(r)F_\varepsilon(r),\ \psi(r)),\ \forall\psi \in D_p^n(r). \tag{42}$$

According to hypothesis we can write $\lim_{\varepsilon\to+0}(F_\varepsilon(r),\ \psi(r)) = (F(r),\ \psi(r))$ and thus from (42) it results

$$\lim_{\varepsilon\to+0}(F_\varepsilon,\ \mu\psi) = \lim_{\varepsilon\to+0}(\mu(r)F_\varepsilon(r),\ \psi(r)) = (F(r)\mu(r),\ \psi(r)),\ \forall\psi\in D_p^n(r),$$

which implies the relation (41). □

Proposition 8. *Let be the function $\alpha(x)\in C^\infty(\mathbb{R}^n)$ with spherical symmetry having the expression $\alpha(x) = \mu\left(\|x\|^2\right)$. Then the function $\mu(r^2)$, $r\in\mathbb{R}$, is the multiplier of the space $D_p^n(r)\subset D(\mathbb{R})$.*

Proof. Let be the test function $\varphi_1\in D(\mathbb{R}^n)$ and $\psi_1\in D_p^n(r)$ its polar transformation given by (16) having the expression

$$T_p^n(\varphi_1) = \psi_1(r) = \int\limits_{S_1(0)}\varphi_1(x)\mathrm{d}S_1 = \int\limits_{S_1(0)}\varphi^*(r,\theta_1,\theta_2,...,\theta_{n-1})\mathrm{d}S_1,$$

where φ_1^* represents the function φ_1 by passing to the polar coordinates given by (3). Because $\alpha(x) = \mu\left(\|x\|^2\right)\in C^\infty(\mathbb{R}^n)$, it means that this is a multiplier of the space $D(\mathbb{R}^n)$ and, consequently, the function $\varphi(x) = \mu\left(\|x\|^2\right)\varphi_1(x)\in D(\mathbb{R}^n)$, is the polar transformation, hence, we have

$$\begin{aligned} T_p^n(\varphi) = \psi(r) = \int\limits_{S_1(0)}\mu(r^2)\varphi_1^*(r,\theta_1,...,\theta_{n-1})\mathrm{d}S_1 = \\ = \mu(r^2)\int\limits_{S_1(0)}\varphi_1^*(r,\theta_1,...,\theta_{n-1})\mathrm{d}S_1 = \mu(r^2)\psi_1(r)\in D_p^n(r). \end{aligned}$$

On then other hand, taking into account [6], if $\varphi_k(x)\overset{D(\mathbb{R}^n)}{\longrightarrow}0$, then $T_p^n(\varphi_k) = \psi_k(r)\overset{D_p^n(r)}{\longrightarrow}0$. Wherefrom it results $\mu\left(\|x\|^2\right)\varphi_k(x)\overset{D(\mathbb{R}^n)}{\longrightarrow}0$ which implies $T_p^n\left(\mu(\|x\|^2)\varphi_k(x)\right) = \mu(r^2)\psi_k(r)\overset{D_p^n(r)}{\longrightarrow}0$. Thus, the function $\mu(r^2)$, $r\in\mathbb{R}$, is a multiplier of the space $D_p^n(r)\subset D(\mathbb{R})$. The proposition is thus proved. □

Remark 1. *The function $\mu(r) = Ar^{2n}$, $n\in\mathbb{N}$, $A\in\mathbb{R}$, $r\in\mathbb{R}$, is a multiplier of the space $D_p^n(r)$, because $\alpha(x) = A\|x\|^{2n}$, $x\in\mathbb{R}^n$, is with spherical symmetry and $\alpha\in\mathrm{C}^\infty(\mathbb{R}^n)$.*

Proposition 9. *Let be the function $f_\varepsilon\in\mathrm{L}_{loc}(\mathbb{R}^n)$, $n\geq 2$, $\varepsilon>0$, with the properties:*

1. *$f_\varepsilon(x) = F_\varepsilon(r)$, $r = \|x\|$;*

2. *$f_\varepsilon(x)\underset{\varepsilon\to0}{\overset{D'(\mathbb{R}^n)}{\longrightarrow}}\delta(x)$.*

Then $\forall p\in\mathbb{N}$ we have

$$r^{2p+n-1}F_\varepsilon(r)H(r)\underset{\varepsilon\to0}{\overset{D_p'^{\,n}}{\longrightarrow}}0. \tag{43}$$

Proof. Because the conditions from the proposition 6 are fulfill and according to (24) we obtain

$$r^{n-1}F_\varepsilon(r)H(r)\underset{\varepsilon\to0}{\overset{D_p'^{\,n}}{\longrightarrow}}T_p^n\delta(x) = \frac{\delta(r)}{|S_1|}. \tag{44}$$

On the other hand, $\mu(r) = r^{2p}$, $r \in \mathbb{R}$, $p \in \mathbb{N}$, is a multiplier of the space $D_p'^{\,n}(r)$ and according to the proposition 7 we can write

$$r^{2p+n-1}F_\varepsilon(r)H(r) \xrightarrow[\varepsilon\to 0]{D_p'^{\,n}} r^{2p}\frac{\delta(r)}{|S_1|} = 0, \tag{45}$$

because $\left(r^{2p}\delta(r),\ \psi(r)\right) = \left(\delta(r),\ r^{2p}\psi(r)\right) = 0$, $\forall\psi \in D_p^n(r)$. □

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