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# Green's Functions in Mathematical Physics

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ABSTRACT. The determination of Green functions for some operators allows the effective writing of solutions to some boundary problems of mathematical physics.

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## 1. INTRODUCTION

The solution of many problems of mathematical physics is related to the construction of Green's function with the help of which the solutions of the boundary-value problems may be determined explicitly and presented in an integral form. Thus, the Green functions are closely connected to the fundamental solution of the linear differential operator corresponding to a given problem.

In the case in which the fundamental solution is a distribution, we have to deal with Green's distributions.

To illustrate the construction of a Green function as well as its connection to the fundamental solution of the operator corresponding to the given boundary-value problem, we shall consider the heat conduction equation for an infinite bar, the generalized Poisson equation and the vibrating string equation.

## 2. HEAT CONDUCTION EQUATION

We shall consider a cylindrical homogeneous and isotropic bar whose lateral surface is isolated from the rest of the medium. We take the symmetry axis of the bar as $Ox$-axis. We shall denote by $u(x,t),\ x \in \mathbb{R},\ t \geq 0$ the temperature of the bar in the point $x$ at the moment $t$, and by $\varrho,\ c,\ k$ the density, the specific heat and the thermal conductivity. We use the Fourier-Newton law regarding the amount of heat flowing in the unit time across the unit cross section of the bar, then, in the absence of internal sources of heat, the heat conduction equation is

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2},\ \ x \in \mathbb{R},\ t > 0, \tag{1}$$

where the constant $a^2$ has the expression $a^2 = k/\varrho c$.

We consider the parabolic equation (1) with the initial condition

$$u(x,0) = \varphi(x),\ \ \varphi \in C^0(\mathbb{R}). \tag{2}$$

The equation (1) with the condition (2) represents the boundary value problem of the heat conduction in an infinite homogeneous bar.

The boundary value problem has unique solution for the given function $\varphi$. The problem solution can be represented in the form

$$u(x,t) = \mathcal{L}(\varphi(x)) = \int_{-\infty}^{\infty} G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi = (\varphi(\xi), G(x,\xi,t))\,, \tag{3}$$

where the kernel of the integral operator $\mathcal{L}$ is $G(x,\xi,t)$ and is called Green function.

To obtain the Green function, we will consider $\varphi(x) = \delta(x - x_0)$, where $x_0 \in \mathbb{R}$ is a parameter. From (3), it results

$$u(x,t) = (\delta(\xi - x_0), G(x,\xi,t)) = (\delta(\xi), G(x,\xi + x_0,t)) = G(x,x_0,t), \tag{4}$$

hence

$$u(x,t) = G(x,x_0,t). \tag{5}$$

Consequently, the Green function $G(x,x_0,t)$ represents the solution of the equation (1) with the condition

$$u(x,0) = \delta(x - x_0). \tag{6}$$

Hence, the Green function $G(x,\xi,t)$ satisfies the equation (1)

$$\frac{\partial G(x,\xi,t)}{\partial t} = \frac{\partial^2 G(x,\xi,t)}{\partial x^2} \tag{7}$$

and the initial condition

$$G(x,\xi,t)|_{t=0} = \delta(x - \xi). \tag{8}$$

Let $E(x,t) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R})$ be the fundamental solution of the equation (1), hence

$$\frac{\partial E}{\partial t} - a^2\frac{\partial^2 E}{\partial x^2} = \delta(x,t) = \delta(x)\times\delta(t). \tag{9}$$

Applying the Fourier transform with respect to *x* to the equation (9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(\alpha,t) - a^2(-\mathrm{i}\alpha)^2\hat{E}(\alpha,t) = \delta(t), \tag{10}$$

where $\hat{E}(\alpha,t) = \mathrm{F}_x[E(x,t)]$.

The equation (11) shows that the distribution $\hat{E}(\alpha,t)$ is the fundamental solution of the operator

$$\frac{\mathrm{d}}{\mathrm{d}t} + a^2\alpha^2. \tag{11}$$

To determine the distribution $\hat{E}(\alpha,t)$, we consider the homogeneous equation

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} + a^2\alpha^2 V(t) = 0, \tag{12}$$

the general solution of which is

$$V(t) = C\mathrm{e}^{-a^2\alpha^2 t}. \tag{13}$$

Imposing the condition $V(0) = 0$, we obtain $C = 1$ and consequently the fundamental solution of the operator (11) is

$$V(t) = H(t)e^{-a^2\alpha^2 t} = \hat{E}(\alpha,t), \tag{14}$$

where $H$ represents the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{15}$$

Applying the inverse Fourier transform, we yield

$$E(x,t) = \mathrm{F}^{-1}\left[\hat{E}(\alpha,t)\right] = \frac{H(t)}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right), \ \ x \in \mathbb{R}, \ \ t \in \mathbb{R}, \tag{16}$$

because $\mathrm{F}^{-1}[\mathrm{e}^{-b^2x^2}] = \sqrt{\frac{\pi}{|b|}}\exp\left(-\frac{x^2}{4b^2}\right)$.

Considering $t$ as parameter, from (14), we obtain, for $t > 0$

$$\mathcal{E}_t(x) = \frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right). \tag{17}$$

We remark that $E_t(x), \ t > 0$ is an element of the test space $\mathcal{S}(\mathbb{R})$ of indefinitely differentiable functions which, for $|x| \to \infty$, approach zero together with all their derivative of any order, faster than any power of $|x|^{-1}$. It satisfies the relation $\int_{\mathbb{R}} E_t(x)\mathrm{d}x = 1$, because $\int_{\mathbb{R}} \mathrm{e}^{-u^2}\mathrm{d}u = \sqrt{\pi}$.

By direct calculation it is verified that $\mathcal{E}_t$ is the solution of the equation (1) and satisfies the initial condition

$$\mathcal{E}_t(x)\big|_{t=+0} = \delta(x). \tag{18}$$

Indeed, taking into account (14), we have

$$\lim_{t\to+0}\mathrm{F}_x\left[E(x,t)\right] = \mathrm{F}_x\left[\lim_{t\to+0}E_t(x)\right] = \lim_{t\to+0}\hat{E}(\alpha,t) = 1 = \mathrm{F}_x[\delta(x)], \tag{19}$$

hence

$$\lim_{t\to+0}\mathcal{E}_t(x) = \delta(x). \tag{20}$$

Taking account of (7), (8) and (17), we obtain

$$\begin{aligned} G(x,\xi,t) &= E_t(x-\xi) = \frac{\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right)}{2a\sqrt{\pi t}}, \ \ t > 0, \\ E(x-\xi) &= H(t)G(x,\xi,t). \end{aligned} \tag{21}$$

These relations show the dependence between the Green function and the fundamental solution of the heat conduction equation in a homogeneous infinitely bar.

Consequently, the solution of the boundary value problem (1), (2) is given by the convolution with respect to $x \in \mathbb{R}$

$$u(x,t) = \mathcal{E}_t(x) * \varphi(x) = \int_{-\infty}^{\infty} E_t(x-\xi)\varphi(\xi)d\xi = \int_{-\infty}^{\infty} G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi, \tag{22}$$

where $\varphi$ is considered to be a bounded function and $\lim\limits_{t\to+0} u(x,t) = \varphi(x)$ in all continuity points of the function $\varphi$.

All the results can be generalized to $\mathbb{R}^n$. Thus, the heat conduction equation in $\mathbb{R}^n$ is

$$\frac{\partial u(x,t)}{\partial t} = a^2\Delta u(x,t), \ \ t > 0, \ \ a > 0, \ \ x \in \mathbb{R}^n, \tag{23}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$ represents the Laplace operator.

The initial condition is

$$u(x,0) = \varphi(x), \quad \varphi \in C^0(\mathbb{R}^n). \tag{24}$$

The Green function $G(x,\xi,t),\ x,\xi \in \mathbb{R}^n,\ t>0$ for the boundary value problem (23), (24) represents the solution of the equation (23) with the initial condition

$$G(x,\xi,t)|_{t=+0} = \delta(x-\xi). \tag{25}$$

Let $\mathcal{E}_t(x),\ x \in \mathbb{R}^n,\ t>0$, be the solution of the equation (23) with the initial condition

$$\mathcal{E}_t(x)|_{t=+0} = \delta(x). \tag{26}$$

Then, the Green function $G(x,\xi,t)$ is given by the relation

$$G(x,\xi,t) = E_t(x-\xi). \tag{27}$$

The boundary value problem (23), (24) is

$$u(x,t) = \mathcal{E}_t(x) * \varphi(x) = \int_{\mathbb{R}^n} \mathcal{E}_t(x-\xi)\varphi(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^n} G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi \tag{28}$$

where $\varphi$ is considered a bounded function and $\lim\limits_{t\to+0} u(x,t) = \varphi(x)$ in all continuity points of the function $\varphi$.

We remark that the solution $\mathcal{E}_t(x)$ of the equation (23) with the condition (26) can be obtained applying the Fourier transform with respect to the variable $x \in \mathbb{R}^n$. Thus, applying the Fourier transform to the equation

$$\frac{\partial \mathcal{E}_t(x)}{\partial t} = a^2 \Delta \mathcal{E}_t(x) \tag{29}$$

and the condition (26), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}_t(\alpha) + a^2 \|\alpha\|^2 \hat{\mathcal{E}}_t(\alpha) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{E}}_t(\alpha)\Big|_{t=+0} = 1, \tag{30}$$

where $\hat{\mathcal{E}}_t(\alpha) = \mathrm{F}\left[\mathcal{E}_t(x)\right](\alpha),\ \ \alpha \in \mathbb{R}^n$.

The solution of the problem (30) is the function

$$\hat{\mathcal{E}}_t(\alpha) = \exp\left(-a^2 \|\alpha\|^2 t\right), \quad t>0. \tag{31}$$

According to the formula

$$\mathrm{F}\left[\exp\left(-b^2 \|x\|^2\right)\right] = \left(\frac{\sqrt{\pi}}{b}\right)^n \exp\left(-\frac{\|\alpha\|^2}{4b^2}\right), \quad b>0, \tag{32}$$

we obtain

$$\begin{aligned}\mathcal{E}_t(x) &= \mathrm{F}^{-1}\left[\hat{\mathcal{E}}_t(\alpha)\right](x) = \mathrm{F}^{-1}\left[\exp(-a^2\|\alpha\|^2 t)\right] \\ &= \frac{1}{(2a\sqrt{\pi t})^n}\exp\left(-\frac{\|x\|^2}{4a^2 t}\right).\end{aligned} \tag{33}$$

Hence, the Green function corresponding to the boundary value problem of the heat conduction in $\mathbb{R}^n$ is

$$G(x,\xi,t) = \mathcal{E}_t(x-\xi) = \frac{1}{(2a\sqrt{\pi t})^n}\exp\left(-\frac{\|x-\xi\|^2}{4a^2 t}\right). \tag{34}$$

The formulae (28), (34) generalize those obtained for the case $n = 1$. Finally, we observe that $\mathcal{E}_t(x) \in \mathcal{S}(\mathbb{R}^n),\ t > 0$, and we have $\int_{\mathbb{R}^n} \mathcal{E}_t(x)\mathrm{d}x = 1$.

Let $E(x,t) \in \mathcal{D}'(\mathbb{R}^{n+1})$ be the fundamental solution of the equation (23), hence of the operator $\partial/\partial t - a^2\Delta$ of heat conduction in $\mathbb{R}^n$. Proceeding as in the case $n = 1$, we can establish the relation

$$E(x,t) = H(t)\mathcal{E}_t(x). \tag{35}$$

Thus, we can write

$$\frac{\partial E(x,t)}{\partial t} - a^2\Delta E(x,t) = \delta(x,t) = \delta(x) \times \delta(t). \tag{36}$$

## 3. Generalized Poisson equation

The generalized Poisson equation is

$$\Delta u(x,y,z) - k^2 u(x,y,z) = f(x,y,z), \tag{37}$$

where $\Delta$ is the Laplace operator, $k = \mathrm{const}$, and $f$ is a distribution from $\mathcal{D}'(\mathbb{R}^3)$. For $k = 0$ we obtain the Poisson equation

$$\Delta u(x,y,z) = f(x,y,z). \tag{38}$$

By definition, the Green function corresponding to the equation (37) represents the solution of the equation

$$\Delta u(x,y,z) - k^2 u(x,y,z) = \delta(x-\xi, y-\eta, z-\tau), \tag{39}$$

where $\xi,\ \eta,\ \tau$ are parameters.

Let $E(x,y,z) \in \mathcal{D}'(\mathbb{R}^3)$ be the fundamental solution of the operator $\Delta - k^2$; hence, we have

$$\Delta E(x,y,z) - k^2 E(x,y,z) = \delta(x,y,z). \tag{40}$$

The Green function $G(x,y,z;\xi,\eta,\tau)$ satisfies the equation

$$\Delta G(x,y,z;\xi,\eta,\tau) - k^2 G(x,y,z;\xi,\eta,\tau) = \delta(x-\xi, y-\eta, z-\tau) \tag{41}$$

and, consequently, the dependence between *G* and *E* is

$$G(x,y,z;\xi,\eta,\tau) = E(x-\xi, y-\eta, z-\tau). \tag{42}$$

This relation shows that the Green function is obtained by a translation of the distribution $E(x,y,z)$ to the point $(\xi,\eta,\tau)$.

Applying the Fourier transform to the equation (40) in $\mathbb{R}^3$, we obtain

$$-(\alpha^2+\beta^2+\gamma^2)\hat{E}(\alpha,\beta,\gamma) - k^2\hat{E}(\alpha,\beta,\gamma) = 1 \tag{43}$$

where $\mathrm{F}\left[E(x,y,z)\right] = \hat{E}(\alpha,\beta,\gamma)$ and $\alpha,\ \beta,\ \gamma$ are real variables.

From (43) it results

$$\hat{E}(\alpha,\beta,\gamma) = -\frac{1}{k^2+\alpha^2+\beta^2+\gamma^2}. \tag{44}$$

We shall denote by $\mathrm{F}_x[\ ],\ \mathrm{F}_y[\ ],\ \mathrm{F}_z[\ ]$ the Fourier transform in $\mathbb{R}$, with respect to the variables $x,\ y,\ z$, respectively; we have

$$\mathrm{F}_x\left[\frac{1}{r}\exp(-kr)\right] = 2K_0\left(\sqrt{\alpha^2+k^2}\sqrt{y^2+z^2}\right), \tag{45}$$

where $K_0$ represents the modified Bessel function of second kind and zero order and $r = \sqrt{x^2+y^2+z^2}$ is the radius vector.

Also, we can write

$$\mathrm{F}_y\left[K_0\left(\sqrt{\alpha^2+k^2}\sqrt{y^2+z^2}\right)\right]=\frac{\pi\exp\left(-|z|\sqrt{\alpha^2+\beta^2+k^2}\right)}{\sqrt{\alpha^2+\beta^2+k^2}}, \tag{46}$$

$$\mathrm{F}_z\left[\frac{\exp\left(-|z|\sqrt{\alpha^2+\beta^2+k^2}\right)}{\sqrt{\alpha^2+\beta^2+k^2}}\right]=\frac{2}{\alpha^2+\beta^2+\gamma^2+k^2}. \tag{47}$$

Because F[ ] represents the Fourier transform in $\mathbb{R}^3$, with respect to all variables, we have

$$\mathrm{F}\left[\frac{1}{r}\exp(-kr)\right]=\frac{4\pi}{\alpha^2+\beta^2+\gamma^2+k^2}, \tag{48}$$

wherefrom

$$\mathrm{F}^{-1}\left[\frac{1}{\alpha^2+\beta^2+\gamma^2+k^2}\right]=\frac{1}{4\pi r}\exp(-kr), \tag{49}$$

*r* being the radius vector.

Taking into account (44), we obtain

$$E(x,y,z)=\mathrm{F}^{-1}\left[\hat{E}(\alpha,\beta,\gamma)\right]=-\frac{1}{4\pi r}\exp(-kr); \tag{50}$$

in accordance with the relation (42), the Green function corresponding to the generalized Poisson equation has the expression

$$G(x,y,z;\xi,\eta,\tau)=-\frac{1}{4\pi\rho}\exp(-k\rho), \tag{51}$$

where

$$\rho=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\tau)^2}. \tag{52}$$

Particularly, for $k=0$, we obtain the Green function corresponding to the equation (38) in the form

$$G(x,y,z;\xi,\eta,\tau)=E_1(x-\xi,y-\eta,z-\tau)=-\frac{1}{4\pi\rho}, \tag{53}$$

where $E_1(x,y,z)=-1/4\pi r$ represents the fundamental solution of the Laplace operator $\mathit{\Delta}$ in $\mathbb{R}^3$, hence

$$\mathit{\Delta} E_1(x,y,z)=\delta(x,y,z). \tag{54}$$

With the help of the Green function (51), the solution of the generalized Poisson equation is

$$u(x,y,z)=E(x,y,z)*f(x,y,z), \tag{55}$$

where $f\in\mathcal{D}'(\mathbb{R}^3)$ is a distribution with compact support.

If *f* is a distribution of function type for which the convolution product $E(x,y,z)*f(x,y,z)$ exists, then, developing the convolution product (55), we obtain

$$\begin{aligned}u(x,y,z)&=\int_{\mathbb{R}^3}E(x-\xi,y-\eta,z-\tau)f(\xi,\eta,\tau)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau\\&=\int_{\mathbb{R}^3}G(x,y,z;\xi,\eta,\tau)f(\xi,\eta,\tau)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\tau=\mathcal{L}\left(f(x,y,z)\right),\end{aligned} \tag{56}$$

where the kernel of the integral operator $\mathcal{L}$ is the Green function.

**Remark 1.** . *The fundamental solution $E(x,y,z) \in \mathcal{D}'(\mathbb{R}^3)$ of the operator $\Delta + k^2$ can be obtained using the derivative rule of the homogeneous functions having at the origin, $x = 0$, a singular point. Then, according with [21], p.126, the function $h(x) = \exp(\mathrm{i}kr)r^{2-n},\ x \in \mathbb{R}^n - \{0\},\ r = \|x\|,\ k \in \mathbb{C},\ n \geq 3$, is locally integrable, and the functions $x_j r^{1-n},\ x_j r^{-n}$ are homogeneous functions. It is shown the relation*

$$\Delta h(x) + k^2 h(x) = -\mathrm{i}k(n-3)\frac{h(x)}{r} - (n-2)S_1\delta(x), \tag{57}$$

*where $S_1 = 2\pi^{n/2}/\Gamma(n/2)$ represents the area of the unit sphere from $\mathbb{R}^n$. Particularly, for $n = 3$, $S_1 = 4\pi$ and considering $k = \mathrm{i}k_1$, we obtain the relation*

$$\Delta h_1(x,y,z) - k_1^2 h_1(x,y,z) = -4\pi\delta(x,y,z), \tag{58}$$

*where*

$$h_1(x,y,z) = \frac{1}{r}\exp(-k_1 r),\ \ r = \sqrt{x^2+y^2+z^2}. \tag{59}$$

*From (58), it results*

$$\Delta\left(-\frac{h_1}{4\pi}\right) - k_1^2\left(-\frac{h_1}{4\pi}\right) = \delta(x,y,z); \tag{60}$$

*hence, the fundamental solution of the operator $\Delta - k_1^2$ is*

$$E(x,y,z) = -\frac{h_1(x,y,z)}{4\pi} = -\frac{1}{4\pi r}\exp(-k_1 r). \tag{61}$$

## 4. Green's function for the vibrating string

For small vibrations of an homogeneous string, in the absence of the external forces, the motion equation of hyperbolic type is

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a^2\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{62}$$

where the constant $a^2$ has the expression $a^2 = T/\varrho$, where $T$ is the tension in the string, and $\varrho$ is the density of the string per unit of length.

In the case of an infinite string the Cauchy problem consists in the determination of the function $u(x,t) \in C^2(\mathbb{R}\times\mathbb{R}_+)$ which satisfies the equation (62), as well as the initial conditions

$$u(x,t)\big|_{t=0} = \varphi(x),\quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \psi(x), \tag{63}$$

where $\varphi,\ \psi \in C^0(\mathbb{R})$.

We remark that the vibrating string (62) as well as the initial condition (63) can be considered in the distributions space. The solution of the Cauchy problem will be thus the distribution $u(x,t) \in \mathcal{D}'(\mathbb{R})$ depending on the parameter $t \geq 0$; $\varphi,\ \psi$ are distributions from $\mathcal{D}'(\mathbb{R})$.

The Green function corresponding to the Cauchy problem (62) and (63) for the vibrating string is $G(x,\xi,t) \in C^2(\mathbb{R}\times\mathbb{R}_+),\ x \in \mathbb{R},\ t \geq 0$, where $\xi \in \mathbb{R}$ is a parameter, satisfying the conditions

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = 0,\ \ x \in \mathbb{R},\ t > 0, \tag{64}$$

$$G(x,\xi,t)|_{t=+0}=0,\quad \frac{\partial}{\partial t}G(x,\xi,t)\bigg|_{t=+0}=\delta(x-\xi). \tag{65}$$

Let $\mathcal{E}_t(x)=\mathcal{E}(x,t)\in C^2(\mathbb{R}\times\mathbb{R}_+)$ be the solution of the equation (62), hence

$$\frac{\partial^2}{\partial t^2}\mathcal{E}_t(x)-a^2\frac{\partial^2}{\partial x^2}\mathcal{E}_t(x)=0, \tag{66}$$

which satisfies the conditions

$$\mathcal{E}_t(x)|_{t=+0}=0,\quad \frac{\partial}{\partial t}\mathcal{E}_t(x)\bigg|_{t=+0}=\delta(x). \tag{67}$$

In this way, between the functions $G(x,\xi,t)$ and $\mathcal{E}_t$ we have

$$g(x,\xi,t)=\mathcal{E}_t(x-\xi). \tag{68}$$

With the help of Green's function, the solution of the Cauchy problem (62) and (63) for the vibrating string can be write in the form

$$\begin{aligned} u(x,t)&=\mathcal{E}_t(x)*\psi(x)+\frac{\partial}{\partial t}\left(\mathcal{E}_t(x)*\varphi(x)\right)\\ &=\int_{\mathbb{R}}\mathcal{E}_t(x-\xi)\psi(\xi)\mathrm{d}\xi+\frac{\partial}{\partial t}\int_{\mathbb{R}}\mathcal{E}_t(x-\xi)\varphi(\xi)\mathrm{d}\xi. \end{aligned} \tag{69}$$

Taking into account (68), we have

$$u(x,t)=\int_{\mathbb{R}}G(x,\xi,t)\psi(\xi)\mathrm{d}\xi+\frac{\partial}{\partial t}\int_{\mathbb{R}}G(x,\xi,t)\varphi(\xi)\mathrm{d}\xi. \tag{70}$$

We shall denote by $\mathrm{P(D)}=\mathrm{P}\left(\partial_t,\partial_x\right)$ the operator corresponding to the vibrating string, namely

$$\mathrm{P(D)}=\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}. \tag{71}$$

Because

$$\mathrm{P(D)}\mathcal{E}_t(x)=\frac{\partial^2\mathcal{E}_t(x)}{\partial t^2}-a^2\frac{\partial^2\mathcal{E}_t(x)}{\partial x^2}=0,$$

on the basis of differentiation rule of the convolution product we obtain

$$\mathrm{P(D)}u(x,t)=\mathrm{P(D)}\mathcal{E}_t(x)*\psi(x)+\frac{\partial}{\partial t}\left[\mathrm{P(D)}\mathcal{E}_t(x)*\varphi(x)\right]=0. \tag{72}$$

From here it results that the two terms of the solution (69), $\mathcal{E}_t(x)*\psi(x)$ and $\partial\left(\mathcal{E}_t(x)*\phi(x)\right)/\partial t$ are solutions of the vibrating string equation (62).

Taking into account the conditions (67) we obtain

$$\begin{aligned} u(x,t)|_{t=+0}=\ &\mathcal{E}_t(x)|_{t=+0}*\psi(x)+\ \frac{\partial\mathcal{E}_t(x)}{\partial t}\bigg|_{t=+0}*\varphi(x)\\ &=0*\psi(x)+\delta(x)*\varphi(x)=\varphi(x), \end{aligned} \tag{73}$$

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t}\bigg|_{t=+0}=\ &\frac{\partial E_t(x)}{\partial t}\bigg|_{t=+0}*\psi(x)+\frac{\partial^2E_t(x)}{\partial t^2}\bigg|_{t=+0}*\varphi(x)\\ &=\delta(x)*\psi(x)+0*\varphi(x)=\psi(x), \end{aligned} \tag{74}$$

because

$$\frac{\partial^2E_t(x)}{\partial t^2}\bigg|_{t=+0}=0.$$

Indeed, from the equation

$$\frac{\partial^2 \mathcal{E}_t(x)}{\partial t^2} - a^2 \frac{\partial^2 \mathcal{E}_t(x)}{\partial x^2} = 0, \tag{75}$$

by passing to the limit, we obtain

$$\lim_{t\to+0} \frac{\partial^2 \mathcal{E}_t(x)}{\partial t^2} = a^2 \frac{\partial^2}{\partial x^2}\left[\lim_{t\to+0} \mathcal{E}_t(x)\right] = a^2 \frac{\partial^2}{\partial x^2}(0) = 0. \tag{76}$$

We shall determine now the expression of the function $\mathcal{E}_t(x),\ x \in \mathbb{R},\ t > 0$.

Applying the Fourier transform with respect to the variable $x \in \mathbb{R}$ to the equation (75) and the conditions (67), we obtain

$$\begin{aligned} &\tfrac{d^2}{dt^2}\hat{\mathcal{E}}_t(\alpha) + a^2\alpha^2\hat{\mathcal{E}}_t(\alpha) = 0, \\ &\hat{\mathcal{E}}_t(\alpha)\Big|_{t=+0} = 0,\ \ \tfrac{d}{dt}\,\hat{\mathcal{E}}_t(\alpha)\Big|_{t=+0} = 1. \end{aligned} \tag{77}$$

The solution of this Cauchy problem is

$$\hat{\mathcal{E}}_t(\alpha) = \frac{\sin(a\alpha t)}{a\alpha}. \tag{78}$$

Applying the inverse Fourier transform, we obtain

$$\mathcal{E}_t(x) = \mathrm{F}_x^{-1}\left[\hat{\mathcal{E}}_t(\alpha)\right] = \begin{cases} \frac{1}{2a}, & |x| \le at, \\ 0, & |x| > at. \end{cases} \tag{79}$$

Indeed, we can write

$$\begin{aligned} \mathrm{F}_x\left[\mathcal{E}_t(x)\right] &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha x}\mathcal{E}_t(x)dx = \frac{1}{2a}\int_{-at}^{at} \mathrm{e}^{\mathrm{i}\alpha x}dx = \frac{1}{2\mathrm{i}\alpha x}\,\mathrm{e}^{\mathrm{i}\alpha x}\big|_{-at}^{at} \\ &= \frac{1}{2\mathrm{i}\alpha a}\left(\mathrm{e}^{\mathrm{i}\alpha ta} - \mathrm{e}^{-\mathrm{i}\alpha ta}\right) = \frac{1}{a\alpha}\sin(at\alpha). \end{aligned} \tag{80}$$

Taking into account (68), we shall obtain for Green's function corresponding to the Cauchy problem of the vibrating spring the expression

$$G(x,\xi,t) = \mathcal{E}_t(x-\xi) = \begin{cases} \frac{1}{2a}, & |x-\xi| \le at \\ 0, & |x-\xi| > at \end{cases} = \begin{cases} \frac{1}{2a}, & \xi \in [x-at, x+at], \\ 0, & \xi \notin [x-at, x+at]. \end{cases} \tag{81}$$

Thus, (70) becomes

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi + \frac{1}{2a}\frac{\partial}{\partial t}\int_{x-at}^{x+at} \varphi(\xi)\mathrm{d}\xi. \tag{82}$$

Applying the Leibniz formula of differentiation of integrals depending on a parameter we obtain

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi, \tag{83}$$

which represents the d'Alembert's formula.

We remark that the formula (69) take place in the case when the equation (62) is considered in the distributions space $\mathcal{D}'(\mathbb{R})$, where $u(x,t)$ is a distribution from $\mathcal{D}'(\mathbb{R})$, $t > 0$ being the parameter, and $\varphi$, $\psi$ from the initial conditions (63) being distributions with compact support.

Let $E(x,t) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R})$ be the fundamental solution of the operator (71), hence

$$\frac{\partial^2 E(x,t)}{\partial t^2} - a^2 \frac{\partial^2 E(x,t)}{\partial x^2} = \delta(x,t) = \delta(x)\times\delta(t). \tag{84}$$

The expression of $E(x,t)$ is

$$E(x,t) = \frac{1}{2a} H\left(at - |x|\right) = \begin{cases} 0, & t < 0, \\ \frac{1}{2a}, & 0 \le |x| \le at, \\ 0, & |x| > at \ge 0, \end{cases}$$

or

$$E(x,t) = H(t)\mathcal{E}_t(x). \tag{85}$$

Indeed, applying the Fourier transform with respect to the variable $x \in \mathbb{R}$, we obtain

$$\frac{d^2}{dt^2}\hat{\mathcal{E}}(\alpha,t) + a^2\alpha^2\hat{\mathcal{E}}(\alpha,t) = \delta(t), \tag{86}$$

where $\hat{E}(\alpha,t) = \mathrm{F}_x\left[E(x,t)\right]$.

This relation shows that the distribution $\hat{E}(\alpha,t)$ is the fundamental solution of the operator $\mathrm{d}^2\big/\mathrm{d}t^2 + a^2\alpha^2$.

But, the fundamental solution of this operator is the distribution

$$\hat{E}(\alpha,t) = H(t)\frac{\sin\left(a\alpha t\right)}{a\alpha}. \tag{87}$$

Applying the inverse Fourier transform we obtain

$$E(x,t) = \mathrm{F}_x^{-1}\left[\hat{E}(\alpha,t)\right] = \frac{1}{2a}H\left(at - |x|\right), \tag{88}$$

this because

$$\begin{aligned} &\mathrm{F}\left[H(R-|x|)\right] = \textstyle\int_{-R}^{R} \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}x = 2\frac{\sin(R\alpha)}{\alpha}, \\ &H(R-|x|) = \begin{cases} 0, & R < |x|, \\ 1, & |x| \le R. \end{cases} \end{aligned} \tag{89}$$

From (85) and (68), it results

$$E(x-\xi,t) = H(t)\mathcal{E}_t(x-\xi) = H(t)G(x,\xi,t), \tag{90}$$

fact which shows the dependence between the fundamental solution $E(x,t)$ of the vibrating spring equation (62) and the Green's function corresponding to Cauchy problem (62), (63).

## References


[1] Schilov, G.E.,*Generalized Functions and Partial Differential Equations*, Gordon and Breach Science Publishers, 1968

[2] Kecs, W.W., *Teoria distribuțiilor și aplicații*, Editura Academiei Române, Bucharest, 2003

[3] Nowacki, W., *Dinamica sistemelor elastice*, Editura Tehnică, București, 1969



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